Addition of Angular Momentum

$$\rightsquigarrow \mathcal{H}_1 = \sum_{J_1} \mathcal{H}_1^{(J_1)}$$

$\vec{J}_1 \qquad |j_1, m_1\rangle \quad m_1 = -j_1, \dots, j_1 \qquad \mathcal{H}_1^{(\vec{J}_1)} = \{ |j_1, m_1\rangle \}$

$\vec{J}_2 \qquad |j_2, m_2\rangle \quad m_2 = -j_2, \dots, j_2 \qquad \mathcal{H}_2^{(\vec{J}_2)} = \{ |j_2, m_2\rangle \}$

Define: $\vec{J} = \vec{J}_1 + \vec{J}_2 = \vec{J}_1 \otimes \mathbb{1} + \mathbb{1} \otimes \vec{J}_2$

acts on vectors in $\mathcal{H}_1^{(J_1)} \otimes \mathcal{H}_2^{(J_2)}$

Uncoupled basis $\quad |j_1, j_2, m_1, m_2\rangle = |j_1, m_1\rangle \otimes |j_2, m_2\rangle$

$$\begin{Bmatrix} J_1^2 \\ J_{1z} \\ J_2^2 \\ J_{2z} \end{Bmatrix} |j_1, m_1\rangle \otimes |j_2, m_2\rangle = \begin{Bmatrix} \hbar^2 j_1(j_1+1) \\ m_1 \hbar \\ \hbar^2 j_2(j_2+1) \\ m_2 \hbar \end{Bmatrix} |j_1, m_1\rangle \otimes |j_2, m_2\rangle$$

Complete set of comuting observables $\rightarrow$ This makes it perfect to use it to define state basis

However we can also think about our in terms of $\vec{J}$ operators:

$$\vec{J}^2 = (\vec{J}_1 \otimes \mathbb{1} + \mathbb{1} \otimes \vec{J}_2)(\vec{J}_1 \otimes \mathbb{1} + \mathbb{1} \otimes \vec{J}_2)$$

$$= J_1^2 \otimes \mathbb{1} + \mathbb{1} \otimes \vec{J}_2 + \sum_k J_{1k} \otimes J_{2k}$$

$$\Rightarrow [J_{1z}, \vec{J}^2] \neq 0 \qquad [J_{2z}, \vec{J}^2] \neq 0 \qquad \text{but } [J_z, \vec{J}^2] = 0$$

another comuting observables: $\{ J_1^2, J_2^2, \vec{J}^2, J_z \}$

natural basis (of eigenvectors) $\quad |j_1, j_2, j, m\rangle \equiv |j, m\rangle$

Coupeling Basis

Both basis are orthonormal $\longrightarrow$ they both come from Hermitian operators

Orthonormal: $\displaystyle\sum_{j_1 j_2} \sum_{m_1 m_2} |j_1 j_2 m_1 m_2\rangle\langle j_1 j_2 m_1 m_2| = \mathbb{1}$ on $\mathcal{H}_1 \otimes \mathcal{H}_2 \rightsquigarrow$ for any given j_1, j_2 or $(\vec{J}_1, \vec{J}_2)$

Anyhow we could also think about on smaller Hilbert spaces where we fixed J_1 and J_2 operators.

$$\sum_{m_1 m_2} |j_1 j_2 m_1 m_2\rangle\langle j_1 j_2 m_1 m_2| = \mathbb{1} \text{ on } \mathcal{H}_1^{(J_1)} \otimes \mathcal{H}_2^{(J_2)}$$

J_1 an J_2 are fixed operators

Since j_1 and j_2 are fixed values we can use the completeness relation to write one basis in terms of the other

$$\Rightarrow \quad |j_1 j_2, jm\rangle = \sum_{m_1 m_2} |j_1 j_2 m_1 m_2\rangle \underbrace{\langle j_1 j_2 m_1 m_2 | j_1 j_2 ; jm\rangle}_{\text{Clebsch-Gordon Coefficients}}$$

$$C(j_1 j_2 j ; m_1, m_2, m)$$

Properties: 1) Vanish if $m \neq m_1 + m_2$

Proof: $J_z = J_{1z} + J_{2z} \longrightarrow (J_z - J_{z1} - J_{z2}) = 0$

acting

$$\langle j_1 j_2, m_1 m_2 | (J_z - J_{z1} - J_{z2}) | j_1 j_2, jm\rangle = 0$$

$$\langle j_1 j_2, m_1 m_2 | \hbar m - \hbar m_1 - \hbar m_2 | j_1 j_2, jm\rangle = 0$$

$$\boxed{(m - m_1 - m_2)\langle j_1 j_2, m_1 m_2 | j_1 j_2 ; jm\rangle = 0}$$

2-) CG coeffs vanish unless $|j_1 - j_2| \leq j \leq j_1 + j_2$ $\rightarrow$ think about $\vec{J}_1, J_2$ as vectors

each j occurs once

Uncoupled basis

$$N_j = (2j_1 + 1)(2j_2 + 1) \text{ \# states}$$

$$N_{\text{coupled}} = \sum_{j=|j_1-j_2|}^{j_1+j_2} (2j+1) \quad \text{Assume } j_1 \geq j_2$$

Assume (2) is true

$$= \sum_{j=0}^{j_1+j_2} (2j+1) - \sum_{j=0}^{j_1-j_2-1} (2j+1)$$

$$= 4j_1 j_2 + 2j_1 + 2j_2 + 1$$

$$= (2j_1 + 1)(2j_2 + 1)$$

consistency

or top states

$$|j_1 j_2, j = j_1 + j_2, m = j_1 + j_2\rangle = |j_1 j_2 ; m = j_1, m_2 = j_2\rangle$$

$$J^2 = J_1^2 + J_2^2 + 2J_1 J_2 = J_1^2 + J_2^2 + 2J_{1z}J_{2z} + J_{1+}J_{2-} + J_{1-}J_{2+}$$

$$J^2 |j_1 j_2, j_1 + j_2, j_1 + j_2\rangle =$$

$$\hbar^2 (j_1(j_1+1) + j_2(j_2+1)) + 2j_1 j_2 + 0 + 0 \,|\ldots\rangle$$

$$= \hbar^2 (j_1 + j_2)(j_1 + j_2 + 1)\,|\ldots\rangle$$

$$|j_1, j_1\rangle \qquad\qquad |j_2, j_2\rangle$$
$$|j_1, j_1-1\rangle \qquad\qquad |j_2, j_2-1\rangle$$
$$|j_1, j_1-2\rangle \qquad \otimes \qquad \vdots$$
$$\vdots$$
$$|j_1, 2j_1-1\rangle \qquad\qquad |j_2, -j_2\rangle$$
$$\vdots$$
$$|j_1, -j_1\rangle$$

$j_1 > j_2$

$\underbrace{\qquad}_{2j_1+1 \text{ states}}$ $\qquad$ $\underbrace{\qquad}_{2(j_2+1) \text{ states}}$

left hand side has $2(j_1 - j_2)$ more states

$m = j_1 + j_2 \to 1$ state

$m = j_1 + j_2 - 1 \to 2$ states

$m = j_1 + j_2 - 2 \to 3$ states

$\vdots$

$m = j_1 - j_2 \to 2j_2 + 1$ states

$m = j_1 - j_2 - 1 \to 2j_2 + 1$ sates $\quad 2(j_1 - j_2)$ rows

$\vdots$

$m = j_2 - j_1 \to 2j_2 + 1$ states

forms separated multiplets $\to$ Different values of $\underline{j_1, j_2}$ and consectvaly $\underline{m}$ have states with diferent values of j. The multiplets are these different states that undergo the same value of $\underline{j}$

$\to$ Is passive of understanding as Some level of "degenerancy" in total angular momentum

3-) CG coeffs can be choosen to be real

4-) Satisfies a recursion relation